Technical Note

MOVING LEAST-SQUARES APPROXIMATION TO BE USED WITH MESHLESS NUMERICAL ANALYSIS METHODS

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Catatan Redaksi:

Meshless Numerical Analysis Method adalah suatu metode analisa numerik yang berkembang dengan pesat sebagai alternatif metode elemen hingga (*Finite Element Method*) yang sudah cukup terkenal. Makalah ini merupakan seri pertama dari dua untuk mengenalkan salah satu shape function yang banyak digunakan untuk *Meshless Numerical Analysis Method*. Makalah seri kedua akan memperkenalkan salah satu *Meshless Numerical Analysis Method* yang dikenal sebagai *Meshless Local Petrov-Galerkin Method*.

INTRODUCTION

The development of the approximate methods for the numerical solution of partial differential equations has attracted the attention of engineers, physicists and mathematicians for a long time. Many of these approximate solution techniques are well-developed and possess much versatility in analyzing complicated phenomena whose behaviors is governed by increasingly complex partial differential equations. Among these approximate methods, the finite element method is one of the most popular. It has been under development for more than 50 years, and its reliability is already well accepted throughout the world.

In recent years, meshless methods have been developed as alternative numerical approaches in efforts to eliminate known drawbacks of the finite element method (FEM). The main objective in developing meshless methods was to eliminate, or at least reduce, the difficulty of meshing and remeshing of complex structural elements. The nature of the various approximation functions employed by meshless methods allows the descretization or redescretization of problem domains by simply adding or deleting nodes where desired. Nodal connectivity to form an element as in FEM method is not needed, only nodal coordinates and their domain of influence (DOI) are necessary to descretize the problem domain.

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Meshless methods may also reduce other problems associated with the FEM, such as solution degradation due to locking and severe element distortion [1].

There are several meshless methods under current development, including the Element-Free Galerkin (EFG) method proposed by Belytschko et al. [2], the Reproducing Kernel Particle Method (RKPM) proposed by Liu et al. [3], Smooth Particle Hydrodynamics (SPH) method proposed by Gingold and Monaghan [4], Meshless Local Petrov-Galerkin (MLPG) method proposed by Atluri et al. [1], and some other methods. The well-establish EFG method, and newly developed MLPG method use shape functions which are derived from moving least-square (MLS) approximation. The main purpose of this paper is to introduce this MLS approximation which will be presented in next section.

MOVING LEAST-SQUARES APPROXIMATION

Given a set of nodes $x_1, x_2,\ldots,x_N$ and a set of nodal values $u_1, u_2,\ldots,u_N$ the original function f:$f(x_i)= u_i$ is to be approximated using no connectivity information. Consider the approximation as a product of polynomial basis function and a set of coefficients as follows:

$$u_{app}(\mathrm{x}) = \mathrm{p}^T(\mathrm{x})\,\mathrm{a}(\mathrm{x}) = \sum_{i=1}^{m} p_i(\mathrm{x}) a_i(\mathrm{x}) \qquad (1)$$

where p, the polynomial basis function, is a vector with the size mx1 (m is the number of polynomial coefficients), and a is a set of coefficients. Examples of polynomial bases and coefficients a are presented in Table 1.

Table 1. Polynomial Bases

1D	$p^T = [1,x,x^2,\ldots,x^n]$
2D	$p^T = [1,x,y,\ldots x^n,y^n]$
3D	$p^T=[1,x,y,z,\ldots,x^n,y^n,z^n]$
	$a^T = [a_1,a_2,\ldots,a_m]$

It should be noted here that the coefficients used in the approximation (a) depend on the location where the original function is approximated, this is different from the approximation coefficients used in FEM which are constant. The task now is to find these coefficients a. Determination of a is achieved by minimizing a weighted square of discrete error of the function u expressed in following term:

$$J = \sum_{I=1}^{N} w_I(\mathbf{x}-\mathbf{x}_I)[u_{app}(\mathbf{x}_I,\mathbf{x})-u_I]^2 \qquad (2)$$

where $w_I(x-x_I)$ is the degree of influence (weight function) of node I to a point x in the problem domain and $u_{app}(\mathbf{x}_I,\mathbf{x}) = \mathbf{p}^T(\mathbf{x}_I)\mathbf{a}(\mathbf{x})$.

Weight function of a point I has a unit value at that point and smoothly decrease as we move further from that point and finally reach zero value at a certain distance (radius) d_{max}, this zero value is kept constant at regions beyond d_{max}. This d_{max} is named the domain of influence (DOI) of that point. Commonly used weight functions are as follows [5]:

- Gaussian Weight Function

$$w(d) = \begin{cases} e^{-(d/0.4)^2} & \text{for } d \le 1, \\ 0 & \text{for } d > 1. \end{cases}$$

- Quartic Spline Weight Function

$$w(d) = \begin{cases} 1-6d^2+8d^3-3d^4 & \text{for } d \le 1, \\ 0 & \text{for } d > 1. \end{cases}$$

where

$$d = d_I / d_{\max}\ ; d_I = |\mathbf{x}-\mathbf{x}_I|.$$

In matrix form, Equation 2 can be rewritten as:

$$J = (\mathbf{Pa}-\mathbf{u})^T\mathbf{W}(\mathbf{Pa}-\mathbf{u}) \qquad (3)$$

where

$$\mathbf{u}^T = (u_1,u_2,\ldots,u_N)$$

$$\mathbf{P} = \begin{bmatrix} p_1(\mathbf{x}_1) & p_2(\mathbf{x}_1) & . & . & . & p_m(\mathbf{x}_1) \\ p_1(\mathbf{x}_2) & p_2(\mathbf{x}_2) & . & . & . & p_m(\mathbf{x}_2) \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ p_1(\mathbf{x}_N) & p_2(\mathbf{x}_N) & . & . & . & p_m(\mathbf{x}_N) \end{bmatrix}.$$

and

$$\mathbf{W}(\mathbf{x}) = \begin{bmatrix} w_1(\mathbf{x}-\mathbf{x}_1) & 0 & . & . & . & 0 \\ 0 & w_2(\mathbf{x}-\mathbf{x}_2) & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . & 0 & w_N(\mathbf{x}-\mathbf{x}_N) \end{bmatrix}.$$

Minimizing Equation 3 with the respect to coefficients a, some expressions below can be obtained.

$$\frac{\partial J}{\partial \mathbf{a}} = \mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{u} = 0 \qquad (4)$$

where

$$\mathbf{A}(\mathbf{x}) = \mathbf{P}^T\mathbf{W}(\mathbf{x})\mathbf{P}\ .$$

$$\mathbf{B}(\mathbf{x}) = \mathbf{P}^T\mathbf{W}(\mathbf{x})$$

Further, the coefficient a($\mathbf{x}$) can be expressed as:

$$\mathbf{a}(\mathbf{x}) = \mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{u} \qquad (5)$$

then the approximation of the original function in Equation 1 can be obtained. Substituting Equation 5 into Equation 1 and rearrange the equation according to nodal value u_I, Equation 1 can be rewritten in this form:

$$u_{app}(\mathbf{x}) = \sum_{I=1}^{N} \Phi_I(\mathbf{x})u_I \qquad (6)$$

where $\Phi_I(\mathbf{x}) = \mathbf{p}^T(\mathbf{x})\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}_I(\mathbf{x})$ is known as the MLS shape function of node I. $B_I(\mathbf{x})$ is the I^{th} column of matrix B($\mathbf{x}$). In application, for example in stress analysis, the degree of freedom to be computed is the displacement. In order to get the stress, the strain which is obtain from displacement derivatives is needed, therefore the derivatives of the shape function should be determined too.

These following equations are formulation to determine the shape functions derivatives:

$$\Phi_{I,x} = (\mathbf{p}^T \mathbf{A}^{-1}\mathbf{B}_I)_{,x}$$
$$= \mathbf{p}^T{}_{,x}\mathbf{A}^{-1}\mathbf{B}_I + \mathbf{p}^T(\mathbf{A}^{-1})_{,x}\mathbf{B}_I + \mathbf{p}^T\mathbf{A}^{-1}\mathbf{B}_{I,x} \tag{7}$$

where

$$\mathbf{B}_{I,x}(\mathbf{x}) = \frac{dw}{d\mathbf{x}}(\mathbf{x}-\mathbf{x}_I)\mathbf{p}(\mathbf{x}_I),$$
$$\mathbf{A}^{-1}{}_{,x} = -\mathbf{A}^{-1}\mathbf{A}_{,x}\mathbf{A}^{-1},$$
$$\mathbf{A}_{,x} = \sum_{I=1}^{n}\frac{dw}{d\mathbf{x}}(\mathbf{x}-\mathbf{x}_I)\mathbf{p}(\mathbf{x}_I)\mathbf{p}^T(\mathbf{x}_I).$$

PLOTS OF WEIGHT FUNCTION AND MLS SHAPE FUNCTION

This section presents examples of plots of weight function and MLS shape function to make the concept clearer. Let's observe a 2D problem domain (1 unit square) defined by uniformly distributed 11x11 nodes (nodal spacing is equal to 0.1 unit in both *x* and *y* directions). Quartic spline weight function with domain of influence 0.4 unit and quadratic polynomial basis function are used in this example.

Small circles in Figure 1 are the nodal coordinates, the black-filled circle is the center node. The big circle is the domain of influence of the center node. Weight function and MLS shape function and their spatial derivatives are plotted at *y*=0.5, so that a section of the function shape along *x* direction can be shown in order to observe the features of this MLS approximation easier. The plots are presented in Figures 2,3,4 and 5.

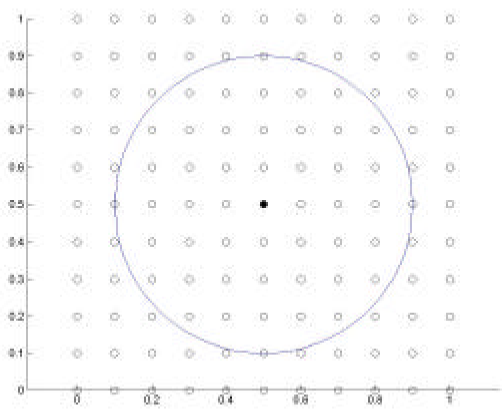

Figure 1. Center node and its domain of influence in a given problem domain descretization.

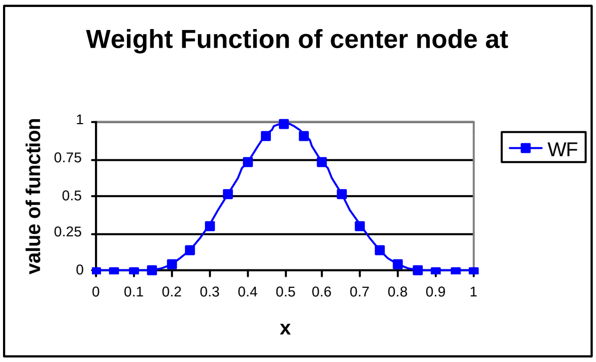

Figure 2. Quartic Spline Weight Function

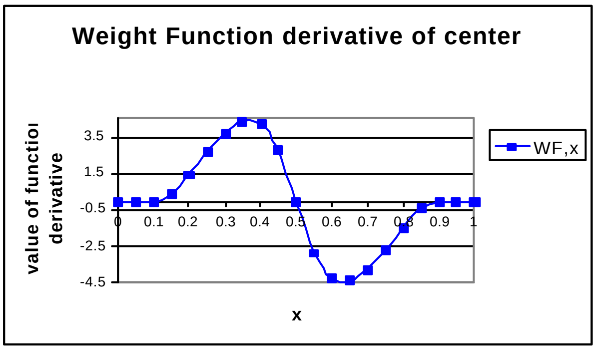

Figure 3. Quartic Spline Weight Function Derivative

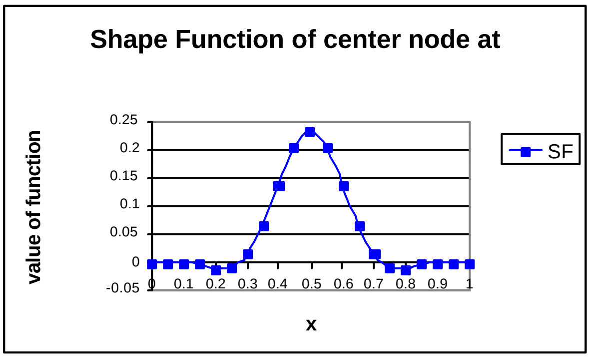

Figure 4. Shape Function of center node

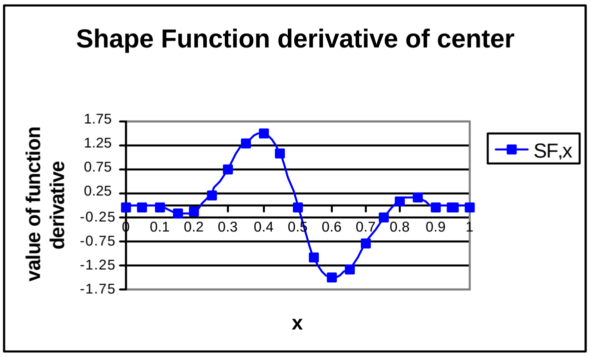

Figure 5. Shape Function Derivative of center node

From Figure 2 through Figure 5, some important things can be noted. First, the weight function is a positive function which has a unit value at corresponding point and decreases to zero at the radius of DOI. MLS shape function does not have unit value at the corresponding node, and it is not necessary equal to zero at any other nodes, which is different from the Lagrange shape function used in FEM. Thus the degree of freedoms solved by MLS approximation are not the real nodal values, they are fictitious. To get the real nodal value or any values inside the problem domain, Equation 6 should be used.

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